Proof that $\frac{3}{2} < (1 + \frac{1}{2n-1})^n \leq 2$

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2018/11/10

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1 Problem statement

Prove $\forall n \in \mathbb{N}_{>0}$:

$$\frac{1}{2} < \left(1 - \frac{1}{2n}\right)^{-n} = \left(1 + \frac{1}{2n-1}\right)^{n} \leq 2$$

1.1 Background

This question was posed as an exercise in the first Calculus class for mathematics students (2nd week). I wasn't able to find a suitable solution even though I spent 10 hours on this problem. Two friends of mine contributed to the solution presented.

1.2 Rationale

Why is this expression of any interest? The expression has the following properties:

$$\lim_{n\to\infty} \left(1 + \frac{1}{2n-1}\right)^{n} = \sqrt{e}$$

So if we consider n at positive infinity, the expression gives the square root of the mathematical constant e.

$$e \approx 2.7183 \qquad \sqrt{e} \approx 1.6487$$

By the way, this is also true, if we remove -1 in the denominator:

$$\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{n} = \sqrt{e}$$

Can you see the similarity to e expressed as limit?

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n} = e$$

2 Proof

2.1 Prerequisite: Bernoulli's inequality

Bernoulli's inequality asserts the following relation between exponential and linear expressions:

Let $r \in \mathbb{N}_{\geq 0}$ and $x \in \mathbb{R}_{\geq -1}$:

$$(1 + x)^r \geq 1 + x \cdot r \tag{1}$$

2.2 Proving the equality

Show that $\left(1-\frac{1}{2n}\right)^{-n} = \left(1+\frac{1}{2n-1}\right)^{n}$:

$$\left(1-\frac{1}{2n}\right)^{-n} = \left(\left(1-\frac{1}{2n}\right)^{-1}\right)^{n} = \left(\frac{2n}{2n-1}\right)^{n} = \left(\frac{2n-1+1}{2n-1}\right)^{n}$$
$$= \left(\frac{2n-1}{2n-1}+\frac{1}{2n-1}\right)^{n} = \left(1+\frac{1}{2n-1}\right)^{n}$$

2.3 Proving the lower bound

Show that $\frac{3}{2} < \left(1+\frac{1}{2n-1}\right)^{n}$ with $n > 0$.

First of all we observe that,

$$\frac{n}{2n-1} \geq \frac{1}{2} \iff 2n \geq 2n-1 \iff 0 \geq -1 \tag{2}$$

The statement $0 \geq -1$ is true, thus $\frac{n}{2n-1} \geq \frac{1}{2}$ is true.

Now consider Bernoulli's inequality (1) and let $x \coloneqq \frac{1}{2n-1}$ and $r \coloneqq n$:

$$\left(1+\frac{1}{2n-1}\right)^{n} \geq 1+\frac{1}{2n-1}\cdot n = 1+\frac{n}{2n-1} \overset{(2)}{\geq} 1+\frac{1}{2} = \frac{3}{2}$$

2.4 Proving the upper bound

Show that $\left(1-\frac{1}{2n}\right)^{-n} \leq 2$ with $n > 0$.

First, we begin with a prerequisite.

$$\begin{aligned}\frac{1}{2}-\frac{1}{4n} \leq 1-\frac{1}{2(n+1)} &\iff -\frac{1}{4n} \leq \frac{1}{2}-\frac{1}{2(n+1)} \\ &\iff \frac{1}{4n} > \frac{1}{2(n+1)}-\frac{1}{2} \\ &\iff \frac{1}{2n} > \frac{1}{n+1}-1 \end{aligned} \tag{3}$$

But is the last statement true? $\frac{1}{n+1}$ is less than 1, because $\frac{1}{2} < 1$ for $n = 1$ and $\frac{1}{n+1} > \frac{1}{n+2}$ (so it is even smaller than $\frac{1}{2}$ for any $n > 1$). So $\frac{1}{n+1} - 1$ is necessarily negative whereas $\frac{1}{2n}$ is positive. Thus $\frac{1}{2n} > \frac{1}{n+1} - 1$ for any $n \geq 1$ and thus $\frac{1}{2} - \frac{1}{4n} \leq 1 - \frac{1}{2(n+1)}$ is true.

Now we rearrange our lower bound statement:

$$\begin{aligned}\left(1-\frac{1}{2n}\right)^{-n} \leq 2 \iff & \frac{1}{\left(1-\frac{1}{2n}\right)^{n}} \leq \frac{2}{1} \\ \iff & \left(1-\frac{1}{2n}\right)^{n} \leq \frac{1}{2} \text{ because both sides are positive } \forall n \\ \iff & \left(\left(1-\frac{1}{2n}\right)^{n}\right)^{-n+1} \geq \left(2^{-1}\right)^{-n+1} \\ \iff & 1-\frac{1}{2n} \geq 2^{-n}\end{aligned}$$

Be aware, that the inequality changes, because $-n+1$ is a non-positive number. Now we prove the last inequality using complete induction over n.

Induction base $n = 1$: $\frac{1}{2} \geq \frac{1}{2}$ is true. Thus the induction base is satisfied.

Induction step $n \to n+1$: Assume $2^{-n} \leq 1 - \frac{1}{2n}$ as induction hypothesis.

$$2^{-(n+1)} = 2^{-n} \cdot 2^{-1} \overset{\substack{\text{induction}\\\text{hypothesis}}}{\leq} \left(1-\frac{1}{2n}\right) \cdot 2^{-1} = \frac{1}{2} - \frac{1}{4n} \overset{(3)}{\leq} 1 - \frac{1}{2(n+1)}$$

In conclusion, the induction step $2^{-(n+1)} \leq 1 - \frac{1}{2(n+1)}$ was proven.

So $1 - \frac{1}{2n} \geq 2^{-n}$ is true and therefore $\left(1-\frac{1}{2n}\right)^{-n} \leq 2$ must be true.