A proof that $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$ with $n \in \mathbb{N}$

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Claim

$$\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1} \qquad n \in \mathbb{N}$$

Preliminaries

Pascal's rule:

$$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$

We derive:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \qquad [k := k-1]$$
$$\binom{n}{k-1} = \binom{n+1}{k} - \binom{n}{k}$$

Proof

Proof. **Induction hypothesis**

$$0 = 0 \qquad \checkmark$$

Induction step Given that $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$, can we show $\sum_{k=0}^{n+1} k\binom{n+1}{k} = (n+1) \cdot 2^{n}$?

$$
\begin{aligned}
\sum_{k=0}^{n+1} k\binom{n+1}{k} &= \sum_{k=0}^{n} k\binom{n+1}{k} + n + 1 \\
&= \sum_{k=0}^{n} k\left[\underbrace{\binom{n+1}{k} - \binom{n}{k}}_{=\binom{n}{k-1}} + \binom{n}{k}\right] + n + 1 \\
&= \sum_{k=0}^{n} k\binom{n}{k-1} + \underbrace{\sum_{k=0}^{n} k\binom{n}{k}}_{n\cdot 2^{n-1}} + n + 1 \\
&= \sum_{k=1}^{n} k\binom{n}{k-1} + n\cdot 2^{n-1} + n + 1
\end{aligned}
$$

Briefly, let's have a look at $\sum_{k=1}^{n} k\binom{n}{k-1}$:

$$
\begin{aligned}
\sum_{k=1}^{n} k\binom{n}{k-1} &= \sum_{k=1}^{n} (k-1+1)\binom{n}{k-1} \\
&= \sum_{k=1}^{n} (k-1)\binom{n}{k-1} + \sum_{k=1}^{n} \binom{n}{k-1} \\
&= \sum_{l=0}^{n-1} l\binom{n}{l} + \sum_{l=0}^{n-1} \binom{n}{l} \\
&= \underbrace{\sum_{l=0}^{n} l\binom{n}{l}}_{=n\cdot 2^{n-1}} - n + \underbrace{\sum_{l=0}^{n} \binom{n}{l}}_{2^n} - 1 \\
&= n\cdot 2^{n-1} + 2^n - (n+1)
\end{aligned}
$$

Followingly,

$$
\begin{aligned}
\sum_{k=0}^{n+1} k\binom{n+1}{k} &= \sum_{k=1}^{n} k\binom{n}{k-1} + n\cdot 2^{n-1} + n + 1 \\
&= n\cdot 2^{n-1} + 2^n - (n+1) + n\cdot 2^{n-1} + (n+1) \\
&= 2\cdot n\cdot 2^{n-1} + 2^n \\
&= n\cdot 2^n + 2^n \\
&= (n+1)\cdot 2^n
\end{aligned}
$$

$\square$