Proof that $2x^2 + 5x - 3$ is continuous in $\mathbb{R}$

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1 Exercise

Prove (using Weierstrass' ϵ-δ-definition of continuity) that f is continuous in $\mathbb{R}$ where

$$f : \mathbb{R} \to \mathbb{R} \qquad x \mapsto 2x^2 + 5x - 3$$

2 Prerequisites

2.1 Triangle inequality

Let a_n be a finite sequence of real values, hence $a_n \in \mathbb{R}\, \forall 0 \leq n \leq r$ with $r < \infty$. Then it holds that

$$\left| \sum_{n=0}^{r} a_n \right| \leq \sum_{n=0}^{r} |\, a_n \,|$$

3 Setting

So technically we need to prove

$$\forall x_0 \in \mathbb{R}\, \forall \varepsilon > 0\, \exists \delta > 0\, \forall x \in \mathbb{R} : |\, x - x_0 \,| < \delta \Rightarrow |\, f(x) - f(x_0) \,| < \varepsilon$$

4 Strategy

To prove this we want to find some δ general enough such that it can be used for all possible values of ε. It is important for those kind of exercises to respect quantifier order. Hence our very general δ is allowed to depend on x_0, ε and (of course) any constants, but *not* x, because it must hold for arbitrary $x \in \mathbb{R}$ as can be read from the setting's expression.

5 $\delta < 1$

Assume that $\delta < 1$ and $|x - x_0| < \delta$. So

$$|x - x_0| < \delta \tag{1}$$
$$|x - x_0| < 1 \tag{2}$$

We can derive:

$$|x| = |x - x_0 + x_0| \tag{3}$$
$$\leq |x - x_0| + |x_0| \tag{4}$$
$$\leq \underbrace{1}_{\text{Inequality } 2} + |x_0| \tag{5}$$

6 Deriving a suitable δ

We will start with the right-hand side expression of our setting. We want to get rid of any x in this expression, because the resulting expression will be used as our general δ. This can be done very easily because of Inequality 2. Furthermore I point out that it is always possible to bring the resulting value closer to ε. So it is always fine to make the expression greater, but making it smaller is not allowed.

$$\left| 2x^2 + 5x - 3 - 2x_0^2 - 5x_0 + 3 \right| \tag{6}$$
$$= \left| 2x^2 + 5x - 2x_0^2 - 5x_0 \right| \tag{7}$$
$$\leq 2|x|^2 + 5|x| - 2|x_0|^2 - 5|x_0| \quad \text{[Triangle ineq.]} \tag{8}$$
$$\leq 2(1 + |x_0|)^2 + 5(1 + |x_0|) - 2|x_0|^2 - 5|x_0| \quad \text{[Inequality 5]} \tag{9}$$
$$= 2(1 + 2|x_0| + |x_0|^2) + 5 + 5|x_0| - 2|x_0|^2 - 5|x_0| \tag{10}$$
$$= 2 + 4|x_0| + 2|x_0|^2 + 5 + 5|x_0| - 2|x_0|^2 - 5|x_0| \tag{11}$$
$$= 2|x_0|^2 - 2|x_0|^2 + 4|x_0| + 5|x_0| - 5|x_0| + 2 + 5 \tag{12}$$
$$= 4|x_0| + 7 \tag{13}$$

7 Selecting δ

Choose $\delta \coloneqq \min\left(1, \frac{\varepsilon}{4|x_0|+7}\right)$. Then it can be proved that any x and x_0 satisfies the equation given in the setting.

8 Proving correctness of δ

It is given that,

$$|\, x - x_0 \,| < \min\left(1, \frac{\varepsilon}{4\,|\, x_0 \,| + 7}\right) \tag{14}$$

$$|\, x - x_0 \,|\,(4\,|\, x_0 \,| + 7) < \min(4\,|\, x_0 \,| + 7, \varepsilon) \tag{15}$$

Because the previous display holds and $\min(4\,|\, x_0 \,| + 7, \varepsilon) \leq \varepsilon$ it certainly holds that,

$$|\, x - x_0 \,|\,(4\,|\, x_0 \,| + 7) < \varepsilon \tag{16}$$

$$|\, x - x_0 \,|\,\left|\, 2x^2 + 5x - 3 - 2x_0^2 - 5x_0 + 3 \,\right| < \varepsilon \tag{17}$$

Because $|\, x - x_0 \,| < 1$ it holds that,

$$\left|\, 2x^2 + 5x - 3 - 2x_0^2 - 5x_0 + 3 \,\right| < \varepsilon \tag{18}$$

$$|\, f(x) - f(x_0) \,| < \varepsilon \tag{19}$$

Our δ always satisfies $f(x) - f(x_0) < \varepsilon$ for all values of $x, x_0 \in \mathbb{R}$. So we have just proved its correctness.