

Proof that x^n is continuous in \mathbb{R}

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1 Exercise

Prove (using Weierstrass' ε - δ -definition of continuity) continuity of f in \mathbb{R} :

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad x \mapsto x^n \quad n \in \mathbb{N}$$

2 Citation

The given proof is an elaboration of the proof idea sketched by John Ma, Math-StackExchange on 18th of Nov 2015.

3 Prerequisites

3.1 Triangle inequality

Let a_n be a finite sequence of real values, hence $a_n \in \mathbb{R} \forall 0 \leq n \leq r$ with $r < \infty$. Then it holds that

$$\left| \sum_{n=0}^r a_n \right| \leq \sum_{n=0}^r |a_n|$$

3.2 Factorization of $a - b$ from $a^n - b^n$

It holds that for given $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$ that,

$$a^n - b^n = (a - b) \cdot \sum_{k=0}^{n-1} a^{n-1-k} b^k \tag{1}$$

$$= (a - b) \cdot (a^{n-1} + a^{n-2}b + \dots + ab^{n-2} + b^{n-1}) \tag{2}$$

4 Setting

Technically we need to prove

$$\forall x_0 \in \mathbb{R} \forall \varepsilon > 0 \exists \delta > 0 \forall x \in \mathbb{R} : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

5 Strategy

To prove this we want to find some δ general enough such that it can be used for all possible values of ε . It is important for those kind of exercises to respect quantifier order. Hence our very general δ is allowed to depend on x_0 , ε and (of course) any constants, but *not* x , because it must hold for arbitrary $x \in \mathbb{R}$ as can be read from the setting's expression.

6 $\delta < 1$

Assume that $\delta < 1$ and $|x - x_0| < \delta$. So

$$|x - x_0| < \delta \quad (3)$$

$$|x - x_0| < 1 \quad (4)$$

We can derive:

$$|x| = |x - x_0 + x_0| \quad (5)$$

$$\leq |x - x_0| + |x_0| \quad (6)$$

$$\leq \underbrace{1}_{\text{Inequality 4}} + |x_0| \quad (7)$$

7 Deriving δ

We look at the right-hand side expression of the setting and reason:

$$|x^n - x_0^n| = \left| (x - x_0) \cdot \sum_{k=0}^{n-1} x^{n-1-k} x_0^k \right| \quad (8)$$

$$= |x - x_0| \cdot \left| \sum_{k=0}^{n-1} x^{n-1-k} x_0^k \right| \quad (9)$$

$$\leq |x - x_0| \cdot \sum_{k=0}^{n-1} |x^{n-1-k} x_0^k| \quad (10)$$

$$= |x - x_0| \cdot \sum_{k=0}^{n-1} |x^{n-1-k}| |x_0^k| \quad (11)$$

$$\leq |x - x_0| \cdot \sum_{k=0}^{n-1} |(1 + x_0)^{n-1-k}| |x_0^k| \quad (12)$$

Let $C := |x - x_0| \cdot \sum_{k=0}^{n-1} |(1 + x_0)^{n-1-k}| |x_0^k|$.

8 Choosing δ

Consider $\delta := \min\left(1, \frac{\varepsilon}{C}\right)$ enforcing $\delta < 1$.

9 Proving correctness of δ

Given

$$|x - x_0| < \min\left(1, \frac{\varepsilon}{C}\right)$$

then

$$|x - x_0| < \min\left(\frac{C}{C}, \frac{\varepsilon}{C}\right) \tag{13}$$

$$|x - x_0| < \frac{1}{C} \min(C, \varepsilon) \tag{14}$$

$$C \cdot |x - x_0| < \min(C, \varepsilon) \tag{15}$$

$$|x^n - x_0^n| < \min(C, \varepsilon) \tag{16}$$

Because $\min(C, \varepsilon) < \varepsilon$ it holds that,

$$|x^n - x_0^n| < \varepsilon \tag{17}$$

$$|f(x) - f(x_0)| < \varepsilon \tag{18}$$