A continuous function in $[0,1]$ with $f(0) = f(1)$ has some ξ such that $f(\xi) = f(\xi + \frac{1}{2})$

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1 Exercise

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function with $f(0) = f(1)$. Prove that there exists some $\xi \in [0, \frac{1}{2}]$ with $f(\xi) = f(\xi + \frac{1}{2})$.

2 Originality

This is based on work by Daniel Smertnig with original notes shared via private conversation in February 2016.

3 Prerequisites

3.1 Intermediate Value Theorem

Let f be a continuous function (there are no jumps/gaps). For any two $f(a)$ and $f(b)$ with $f(a) < f(b)$, there exists some value ξ such that

$$\forall y \in [f(a), f(b)] \exists \xi : f(\xi) = y \land f(a) \leq y \leq f(b)$$

3.2 Algebraic Continuity Theorem

One of the statements of the Algebraic Continuity Theorem is that

Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous in $c \in A$. Then $f(x) + g(x)$ is continuous at c.

4 Proof

4.1 Auxiliary function $h(x)$ and its continuity

Consider the function $h : [0, \frac{1}{2}] \to \mathbb{R}$ with $h(x) = f(x) - f(x + \frac{1}{2})$. Consider the Algebraic Continuity Theorem for entire $\mathbb{R}$. It follows that h is continuous.

4.2 Boundary values of h

$$\begin{aligned} h(0) &= f(0) - f(\tfrac{1}{2}) \\ h(\tfrac{1}{2}) &= f(\tfrac{1}{2}) - f(1) &= f(\tfrac{1}{2}) - f(0) \end{aligned}$$

$$\Rightarrow h\left(\frac{1}{2}\right) = (-1)\left(-f\left(\frac{1}{2}\right) + f(0)\right) = -h(0)$$

4.3 Case distinction

Consider $h(0) \leq h\left(\frac{1}{2}\right)$. Then

$$f(0) - f\left(\frac{1}{2}\right) \leq f\left(\frac{1}{2}\right) - f(0)$$
$$2f(0) \leq 2f\left(\frac{1}{2}\right)$$
$$f(0) \leq f\left(\frac{1}{2}\right)$$

$$\begin{aligned} 0 &\leq f\left(\tfrac{1}{2}\right) - f(0) & f(0) - f\left(\tfrac{1}{2}\right) &\leq 0 \\ 0 &\leq h\left(\tfrac{1}{2}\right) & h(0) &\leq 0 \end{aligned}$$

Analogously, consider $h(0) \geq h\left(\frac{1}{2}\right)$. Then

$$\begin{aligned} 0 &\geq f\left(\tfrac{1}{2}\right) - f(0) & f(0) - f\left(\tfrac{1}{2}\right) &\geq 0 \\ 0 &\geq h\left(\tfrac{1}{2}\right) & h(0) &\geq 0 \end{aligned}$$

In conclusion,

$$h(0) \leq h\left(\frac{1}{2}\right) \Rightarrow h(0) \leq 0 \leq h\left(\frac{1}{2}\right)$$
$$h(0) \geq h\left(\frac{1}{2}\right) \Rightarrow h\left(\frac{1}{2}\right) \leq 0 \leq h(0)$$

As a single-line statement, we have

$$\min\left(h(0), h\left(\frac{1}{2}\right)\right) \leq 0 \leq \max\left(h(0), h\left(\frac{1}{2}\right)\right)$$

4.4 Application of the Intermediate Value Theorem

You can directly apply the Intermediate Value Theorem to the single-line statement, but it is more easy to recognize it with the case distinction.

- So consider $h(0) \leq h\left(\frac{1}{2}\right)$, then the IVT holds:

$$\forall y \in \left[h(0), h\left(\frac{1}{2}\right)\right] \exists \xi : h(\xi) = y \wedge h(0) \leq y \leq h\left(\frac{1}{2}\right)$$

Consider $y = 0$, then

$$\exists \xi : h(\xi) = 0 \wedge h(0) \leq 0 \leq h\left(\frac{1}{2}\right)$$

So there exists some ξ such that $h(\xi) = 0$.

- Analogously consider $h(0) \geq h\left(\frac{1}{2}\right)$, then the IVT holds as well:

$$\forall y \in \left[h\left(\frac{1}{2}\right), h(0)\right] \exists \xi : h(\xi) = y \wedge h\left(\frac{1}{2}\right) \leq y \leq h(0)$$

  Consider $y = 0$, then

$$\exists \xi : h(\xi) = 0 \wedge h\left(\frac{1}{2}\right) \leq 0 \leq h(0)$$

  So there exists some ξ such that $h(\xi) = 0$.

4.5 Conclusion

So in any possible case it holds that there exists some $\xi \in [0, \frac{1}{2}]$ such that $h(\xi) = 0$.

$$\begin{aligned} h(\xi) &= 0 \\ f(\xi) - f\left(\xi + \frac{1}{2}\right) &= 0 \\ f(\xi) &= f\left(\xi + \frac{1}{2}\right) \end{aligned}$$