

About the kernel of vector spaces

Lukas Prokop

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1 Claim

Let V be a vector space over field K . Let $g \in \text{Hom}(V, V)$. Show that

1. $\text{kernel}(g) \subseteq \text{kernel}(g \circ g)$.
2. If $\text{kernel}(g) = \text{kernel}(g \circ g)$, then also $\text{kernel}(g \circ g) = \text{kernel}(g \circ g \circ g)$.

2 Preliminaries

2.1 Homomorphism property

Let $f \in \text{Hom}(V, V)$ and $a, b \in V$. Then f satisfies,

$$f(a + b) = f(a) + f(b) \tag{1}$$

where $f(a), f(b) \in V$.

2.2 Definition of vector space's kernel

$$\text{kernel}(g) := \{v \in V \mid f(v) = 0\}$$

2.3 Mapping of the zero element

Let $f : V \rightarrow W$ be a mapping satisfying Equation 1.

$$f(a) = f(a) \tag{2}$$

$$f(a + 0_V) = f(a) \tag{3}$$

$$f(a) + f(0_V) = f(a) \tag{4}$$

$$f(a) + f(0_V) = f(a) + 0_W \tag{5}$$

$$f(0_V) = 0_W \tag{6}$$

So the zero element of V must necessarily map to the zero element of W .

3 Proof 1: $\text{kernel}(g) \subseteq \text{kernel}(g \circ g)$

Let v be an element of $\text{kernel}(g)$. So v satisfies

$$\begin{aligned}g(v) &= 0 \\g(g(v)) &= g(0)\end{aligned}$$

Because of Equation 6, it holds that $g(0) = 0$.

$$g(g(v)) = 0$$

v is a general element of $\text{kernel}(g)$ and satisfies the properties of $\text{kernel}(g \circ g)$. So any element of $\text{kernel}(g)$ is an element of $\text{kernel}(g \circ g)$. Followingly $\text{kernel}(g) \subseteq \text{kernel}(g \circ g)$ holds.

4 Proof 2: $\text{kernel}(g) = \text{kernel}(g \circ g) \Rightarrow \text{kernel}(g \circ g) = \text{kernel}(g \circ g \circ g)$

Let $v \in \text{kernel}(g)$. So $g(v) = 0$ and because $\text{kernel}(g) = \text{kernel}(g \circ g)$ it also satisfies $g(g(v)) = 0$.

$$g(v) = 0 = g(g(v)) \Rightarrow g(v) = g(g(v)) \quad (7)$$

$$v \in \text{kernel}(g \circ g) : g(g(v)) = 0 \xrightarrow{\text{Equation 7}} g(g(g(v))) = 0 \quad (8)$$

$$\Rightarrow \text{kernel}(g \circ g) \subseteq \text{kernel}(g \circ g \circ g) \quad (9)$$

$$w \in \text{kernel}(g \circ g \circ g) : g(g(g(w))) = 0 \xrightarrow{\text{Equation 7}} g(g(w)) = 0 \quad (10)$$

$$\Rightarrow \text{kernel}(g \circ g \circ g) \subseteq \text{kernel}(g \circ g) \quad (11)$$

From Equation 9 and 11 it follows that $\text{kernel}(g \circ g) = \text{kernel}(g \circ g \circ g)$.