Combinatorics

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These are mad lecture notes. Some of them where taken on hand-written sheets and some having been put directly onto the handout. All the others can be found in this document. Also I mixed up English and german.

0.1 Literature

- Richard Stanley, Enumerate Combinatorics I
- Martin Aigner, A Course in Enumeration

1 Elementary combinatorics

1.1 Permutations

Notation:

$$S_n = \text{ set of permutations of n elements}$$

$$S_n = \left\{ \underbrace{\Pi}_{\text{permutation}} : \{1,\dots,n\} \underbrace{\rightarrow}_{\text{mapping}} \{1,\dots,n\} \,|\Pi \text{ is bijective} \right\}$$

and $[k;l] = \{k, k+1, \dots, l\}$ with $k \leq l, k, l \in \mathbb{Z}$.

$$[1,n] = [n] = \{1,\dots,n\}$$

$$\pi : 1 \rightarrow 4, 2 \rightarrow 3, 3 \rightarrow 1, 4 \rightarrow 5, 5 \rightarrow 2$$

As a cycle:

(1 4 5 2 3)	cycle representation
4,3,1,5,2	word representation

$$(12)(34) = (34)(12)$$
$$(123) = (231)$$

Permutations of disjoint cycles are equivalent. Cycles in disjoint representation can be swapped.

$$(1253)(46789)$$
$$(5312)(94678)$$

This is the standard representation.

Rule. Greatest number first. Cycles shall be ordered by the left-most number ascending.

Without parentheses:

$$5312\ 94678$$

5,9 are from left to right maxima.

Definition 1.1. (cycle type) A cycle type $(c_1, \ldots, c_n)$ of a permutation π with c_j is the number of j-cycles (cycles of length j) in the cycle representation of π (disjoint).

$$(5312\ 94678) \text{ has cycle type } (0, 0, 0, 1, 1)$$

2 Cycles

Proposition 1.1. (permutations of a cycle type) $(c_1, \ldots, c_n) \in \mathbb{N}_0^n$ is a fixed cycle type.

The number of permutations $\pi \in S_n$ of this given cycle type is:

$$\frac{n!}{1^{c_1} c_1! 2^{c_2} c_2! \ldots n^{c_n} c_n!}$$

Proof.

- There are $n!$ possibilities to distribute $1, \ldots, n$ to the position.
- Swapping of cycles of same length corresponds dividing by $c_j!$.
- There are j cyclic permutation within a j-cycle. Division by j^{c_j}.

2.1 Stirling cycle numbers

Definition 1.2. (as defined by Don Knuth) The number of permutations $\pi \in S_n$ with exactly k cycles is called stirling number of first order.

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

Proposition 1.2. (recursive definition)

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \qquad \begin{bmatrix} 0 \\ k \end{bmatrix} = 0 \qquad \begin{bmatrix} n \\ 0 \end{bmatrix} = 0 \qquad k > 0, n > 0$$

n	0	1	2	3
0	1	0	0	0
1	0	1	0	0
2	0	1	1	0
3	0	2	3	1
	0	6	11	1

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix}$$

Proof. We add an element n:

1. n is an own cycle: (n) thus $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ possibilities.
2. n is added to one of the k cycles. There are $n-1$ positions and $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ arrangements of the $n-1$ numbers with k cycles.

$$\binom{u}{k}, \binom{0}{0} = 1, \binom{0}{k} = 0, \binom{11}{0} = 0, \binom{11}{k} = \binom{n-1}{k-1} \perp (n-1) \binom{n-1}{k}$$

Proposition 1.3.

$$\sum_{k=0}^{\infty} \binom{u}{k} x^k = x(x+1)\ldots(x+n-1) = x^{\overline{n}} \qquad \text{“ascending factorial”}$$

Approach for proof.

- Induction
- Evaluate

3 points are necessary to describe a polynomial unambiguous.

Proof. (induction with recursion)

$$f(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

Base with $n = 0$

$$\binom{0}{0} x^0 = 1 = \underbrace{1}_{\text{empty product}}$$

For $x = 0$ we get an empty product in the formula. Therefore it is by definition 1.

Induction step $n - 1 \to n$:

$$\begin{aligned}
f(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k &= \sum_{k=1}^{\infty} \binom{n}{k} x^k \\
&= \sum_{k=1}^{\infty} \left(\binom{n-1}{k-1} + (n-1) \binom{n-1}{k} \right) x^k \\
&= x \sum_{k=1}^{\infty} \binom{n-1}{k-1} x^{k-1} + (n-1) \sum_{k=1}^{\infty} \binom{n-1}{k} x^k \\
&= x \sum_{k=1}^{\infty} \binom{n-1}{k-1} x^{k-1} + (n-1) f_{n-1}(x) \\
&= x \sum_{k=0}^{\infty} \binom{n-1}{k} x^k + (n-1) f_{n-1}(x) \\
&= x f_{n-1}(x) + (n-1) f_{n-1}(x) \\
&= f_{n-1}(x) \cdot (x + n - 1) \\
&= x \cdot (x+1) \cdot \ldots \cdot (x+n-2) \cdot (x+n-1) = x^{\overline{n}}
\end{aligned}$$

2.2 Second proof: polynomial method

Script reference. Seite 5

$$\sum \binom{n}{k} x^k = x(x+1)\dots(x+n-1)$$

Es reicht dies für n + 1 viele Werte zu zeigen. $x \in \mathbb{N}$.

$$\sum_{k\geq 0} \binom{n}{k} x^k = \sum_{\pi \in S_n} x^{\text{Zahl der Zyklen von } \pi}$$

Wir zählen die (π, f) mit f : Zyklen $\to [x]$ wobei x eine Abkürzung für $\{1, \dots, x\}$ mit $x \in \mathbb{N}$ ist.

$$x(x+1)\dots(x+n-1)$$

$(a_1, \dots, a_n)$ $in\mathbb{Z}^n\cap$ Trapez ist $(x)(x+1)\dots(x+n-1)$.

- Falls a_i im Rechteck liegt $(a_i \leq x-1)$, beginne einen neuen Zyklus mit i f(Zyklus) $= a_i$
- Falls a_i nicht im Rechteckt liegt $(a_i = x-1+k, 1 \leq k \leq n-1)$, platziere i so, dass genau k Zahlen größer als i links von i sind.

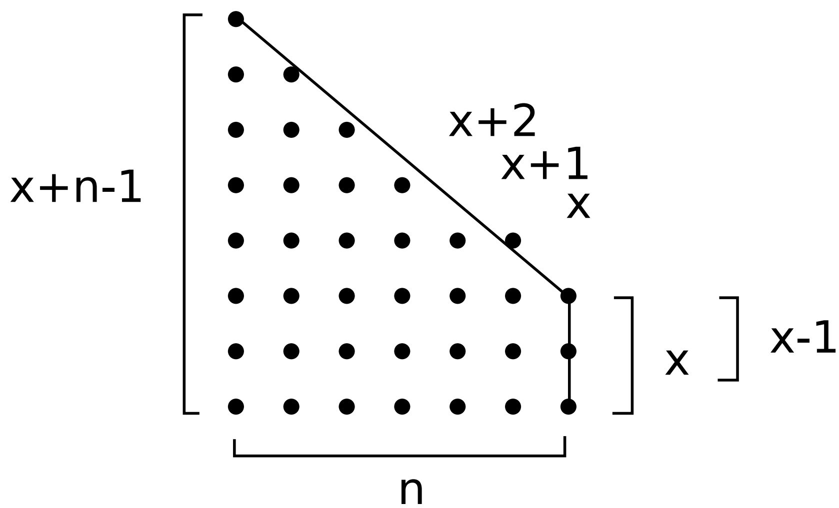

Abbildung 1: Polynommethode

2.3 Beispiel, $n = 9$, $x = 4$

$$(a_1, a_2, \ldots, a_9) = (4, 8, 5, 0, 7, 5, 2, 4, 1)$$

$$n = 9 : a_9 = 1 \in \text{Rechteck}$$
$$n - 1 = 8 : a_8 = 4, 4 \leq x - 1 = 3 \Rightarrow \text{im Rechteck}$$
$$4 = 3 + 1 \Rightarrow k = 1(98)$$
$$n - 2 = 7 : a_7 = 2 \in \text{Rechteck}, (7)(98)$$
$$n - 3 = 6 : a_6 = 5 \notin \text{Rechteck}, 5 = 3 + 2, k = 2, (7)(968)$$
$$\ldots$$
$$(41)(73)(96285)$$
$$f(41) = a_4 + 1$$
$$f(73) = a_7 + 1 = 3$$

$$f(96285) = a_9 + 1 = 2$$

Idee:

$$x(x+1)\dots(x+n-1)$$

Kombinatorisch interpretieren als $(a_1, \dots, a_n)$-Folgen und bijektiv mit Permutationen verhindern.

Spezialfall $x = 1$**:** Permutationen von $S_n : [0, n-1]x[0, n-2], \dots, [0, 0]$.

Die Anzahl der Zyklen entspricht auf dieser Seite der Anzahl der „0" im n-Tupel.

3 Beispiel

Definition 1.3. (Inversionen) Sei $\pi = a_i, \dots, a_n \in S_n$ (Wortdarstellung). Falls $a_i < a_j$ für $i < j$, dann heißt (i, j) Inversion (oder Fehlstellung).

$$i(\pi) = \text{Anzahl der Inversion}$$

Für $(a_i, \dots, a_n)$ wähle $\pi \in S_n$ sodass a_i die Anzahl der Zahlen größer i (links von i) in der Wortdarstellung von π ist.

Beispiel: $n = 9, (a_1, \dots, a_9) = (1, 5, 2, 0, 4, 2, 0, 1, 0)$ (Auflistung der Inversionen). Bei $\pi = (41)(73)(96285)$ erkennen wir beispielweise, dass für 7 2 Inversionen vorliegen.

4 Erzeugende Funktionen

Eine Potenzreihe, deren Koeffizienten so gesetzt sind, dass sie die gewünschten Objekte zählen.

Lemma 1.4 (Erzeugende Funktionen von Inversionen)

$$\sum_{k\geq 0} \left| \{\pi \in S_n : i(\pi) = k| \, q^k = \sum_{\pi \in S_n} q^{i(\pi)} \right\} = [n]_q!$$

Notation. $[k]_q = 1 + q + \ldots + q^{k-1} = \frac{q^k - 1}{q-1}$.

$$[n]_q! = [n]_q[n-1]q \ldots [1]_q$$

4.1 Beweis.

(Herleitung der Faktorielle)

$$\begin{aligned}\sum_{\pi \ inS_n} q^{i(\pi)} &= \sum_{(a_1,a_2,\ldots,a_n)\in[0,n-1]\times\ldots\times[0,0]} q^{a_1+a_2+\ldots+a_n} \\ &= \left(\sum_{a_1=0}^{n-1} q^{a_1}\right) \cdot \left(\sum_{a_2=0}^{n-2} q^{a_2} \cdot \ldots\right) \cdot \ldots \cdot \left(\sum_{a_n=0}^{0} q^{a_n}\right) \cdot \ldots \cdot \\ &= (1 + q + \ldots + q^{n-1})(1 + \ldots + q^{n-2}) \ldots (1+q) \cdot 1 \\ &= [n]_q \cdot [n-1]_q \cdot \ldots \cdot [1]_q \\ &= [n]_q!\end{aligned}$$

4.2 1.3 q-Analog q

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$
$$\text{für } q \to 1 : [k]_q = k, [n]_q! = n!, \binom{n}{a}_q = \binom{n}{k}$$

4.3 Property 1.5

Sei $\mathbb{F}_q$ ein endlicher Körper, dann ist die Anzahl der k-dimensionalen Untervektorräume von $\mathbb{F}_q^n$ ist gleich $\binom{n}{k}_q$.

Beweis. (Prinzip der doppelten Abzählung)

$$N(n,k) = \left\{(v_1, \dots, v_k) | v_1, \dots, v_k \quad \text{linear unabhängig in } \mathbb{F}_q^n\right\}$$

Dabei $N(n,0) = 1$. $N(n,1) = q^n - 1$. $N(n,2) = (q^n - 1)(q^n - q)$. Dieses Prinzip setzt sich fort. $N(n,k) = (q^n - q)(q^n - q)(q^n - q^2) \dots (q^n - q^{k-1})$.

Sei W ein Unterraum von $\dim k$, von $\mathbb{F}_q^n$. Wir zählen die geordneten Basen von W.

Für q_1: $q^k - 1$
Für q_2 und q_1 zusammen: $(q^{k-1})(q^k - q)$
Für q_1 bis q_k zusammen: $(q^{k-1}) \dots (q^k - q^{k-1})$

$$N(n,k) = |\left\{V \subseteq \mathbb{F}_q^n, \dim V = k\right\}| \cdot (q^k - 1) \dots (q^k - q^{k-1})$$
$$|\left\{V \subseteq \mathbb{F}_q^n : \dim V = k\right\}| = \frac{N(n,k)}{(q^k - 1) \cdot \dots \cdot (q^k - q^k - 1)}$$
$$\dots$$
$$= \frac{[n]_q \dots [n-k+1]_q}{[k]_q \dots [1]_q} = \binom{n}{k}_q$$

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(Nicht mehr alles mitgeschrieben. Skriptum verfügbar.)

Topic. Stirling-Partitions-Zahlen **Script reference.** Definition 1.4 und Definition 1.5

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \cdot \left\{ {n-1 \atop k} \right\}$$

n liegt in einer der k Teilmengen mit $n-1$ Elementen.

$$x^{\underline{k}} = x(x-1)\dots(x-k+1)$$

Beweis: Bei einer Abbildung $f : [n] \mapsto [x]$ kann jede Zahl von $[n]$ auf x Elemente abgebildet werden. Daher haben wir insgesamt x^n Abbildungen. $a \sim b \Leftrightarrow f(a) = f(b)$.

Für jede Partition von $[n]$ in k Mengen gibt es $x(x-1)\dots(x-k+1) = x^{\overline{k}}$ verschiedene Abbildungen.

$$\sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\overline{n}}$$

$$\sum_{k \geq 0} \begin{bmatrix} n \\ k \end{bmatrix} x^{\underline{k}} = x^n$$

$x^{\overline{k}}$ ist die steigende Faktorielle

6 Siebmethoden

In einem partially ordered set müssen nicht alle Elemente miteinander vergle-ichbar sein.

Die Potenzmenge einer endlichen Menge ist endlich.

Script reference. Definition 2.4

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Incidence algebra of poset P:

$$I(P) = \{f : \text{intervals}(P) \to \mathbb{C}\}$$

3 operations: $+, \cdot, *$

Unitary means $\exists$ an one-value.

$$(f+g)*(\tilde{f}+\tilde{g}) = f*\tilde{f} + g*g + f*\tilde{g} + g\tilde{g}$$
$$(\alpha f)*(\beta g) = \alpha\beta(f*g)$$
$$(f*\delta)(x,y) = \sum_{z\in[x,y]} f(x,z)\delta(z,y) = \sum_{z\in[x,y],z=y} f(x,z) = f(x,y)$$

Script reference. Proposition 2.2

In group theory left- and right-inverse are not relevant. Here it is.

Abbreviation. TFAE = The following are equivalent.

Script reference. Definition 2.5 Zeta function is somehow related to the Riemann Zeta function.

Script reference. differences calculus

$$0^3, 1^3, 2^3, 3^3, 4^3, 5^3$$
$$0, 1, 8, 27, 64, 125$$
$$1, 7, 19, 37, 61, 91 \text{(differences)}$$
$$6, 12, 18, 24, 30$$
$$6, 6, 6$$
$$0, 0, 0$$

Script reference. page 17 Sum corresponds (approximately) integration.

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$$\simeq \qquad \text{isomorph}$$

Eulersche ϕ-Funktion.

$$A \mathbin{\dot\cup} B \Leftrightarrow \text{Union of } \textit{disjoint} \text{ sets A and B}$$

Script reference. page 21

$$A = \{1\}, B = \{2\}, C = \{3\}$$
$$|u_1| + |u_2| + |u_3|$$
$$A = \{1,2\}, B = \{1,3\}, C = \{2,3\}$$
$$-|u_1 \cap u_2| - |u_1 \cap u_3| - |u_2 \cap u_3|$$
$$A = \{1,2,3\}$$
$$+|u_1 \cap u_2 \cap u_3|$$

Script reference. page 23

Dimension equation („Dimensionsformel"):

$$\dim U + \dim a - \dim U \cap a$$

$$\binom{k}{k-1}_q = \frac{[k]_q[k-1]_q[k-2]_q \dots [1]_q}{[k-1]_q \dots [1]_q} = [k]_q = 1 + q + q^2 + \dots + q^{k-1} = \frac{q^k - 1}{q - 1}$$

8.1 Involution principle

Script reference. page 24

S^+ set of high numbers added to the set artificially. S^- set of low numbers added to the set artificially.

$^+ \cup S_0$ is the set of all paths from $(0,0)$ to $(2n,0)$ without constraints: $\binom{2n}{n}$.

Topic. Vandermonde-Determinant

$$\det \pi \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ 1 & x_3 & \dots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix} = \prod_{i \leq i < j \leq n}$$

$$\begin{pmatrix} 1 & x_1 & x_1^2 \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 \\ \vdots & \ddots & \vdots \\ 0 & x_n - x_1 & x_n^2 - x_n^2 \end{pmatrix}$$

Laplace formula (dt. ,,Laplace'scher Entwicklungssatz").

Summary

For a group G:

$$G_X = \{g \in G | gx = x\} \quad \text{(stabilizer)}$$
$$X_g = \{x \in X | gx = x\} \quad \text{(fixed points)}$$

Orbit is an equivalence class:

$$x \sim_G y \Leftrightarrow \exists g \in G : gx = y$$

$$F = \{f | \mathbb{N} \mapsto \mathbb{R}\}$$
$$gf = f \circ g^{-1} \qquad gf(x) = f(g^{-1}(x))$$

Script reference. page 32

8.2 Rotation of a cube

The rotation group of cubes contains 24 elements. 1 corner mapped to 8 corners. 2 are mapped to 3 possible adjacent ones. $3 \cdot 8 = 24$. X are the vertices

$$|\text{orbit of } 1| = 8$$
$$|G_1| = 3$$

An edge has two mapping. Either itself or the vertices are swapped. X are the edges.

$$|\text{stabilizer of edge}| = 2$$
$$|\text{orbit of an edge}| = 12$$

$|G| = 2 \cdot 12 = 24$.

X are the surfaces.

8.2.1 Listing of all rotations

Rotation axis?

1. identity (0°)
2. rotation along space diagonal ($2 \to 3 \to 4 \to 2$) (120°). Four rotation axes: 4 rotations.
3. rotation along space diagonal (240°). Four rotation axes: 4 rotations.
4. rotation along axis orthogonal to a surface (90°). Three rotation axes: 3 rotations.
5. rotation along axis orthogonal to a surface (180°). Three rotation axes: 3 rotations.

6. rotation along axis orthogonal to a surface (270°). Three rotation axes: 3 rotations.

7. rotation along axis intersecting midpoints of edges which are on the other side of the cube (270°). Six rotation axes: 6 rotations.

Sum: 24.

48 rotation reflections.

$S_4 \cdot C_2$.

Topic. Stigler's Law

Topic. Burnsides Lemma

Handout. (M. Aigner "A course in enumeration") Given a chess board with 3×3 fields. How many rooks can be placed without attacking each other? 6 possibilities for 3 rooks.

Script reference. page 40

Theorem. For every defined number of colors r and for all numbers $s_1, \ldots, s_r$: $\exists R = \mathcal{R}(r; s_1, \ldots, s_r) < \infty$ such that a K_R contains a monochromatic K_{s_i} colored with i (for at least one i).

Sketch of proof. 3 colors = sum up 2 colors = 2 colors. Apply ramsey theorem for 2 colors iteratively.

$$\mathcal{R}(r = 1; s_1, s_2, \ldots, s_{r+1}) \leq \mathcal{R}(2, \mathcal{R}(r; s_1, \ldots, s_r), s_{r+1})$$

Open questions in Ramsey-theory (problem complexity grows extraordinary fast):

$$R(3,4) = 9$$

$$R(4,4)$$

$$R(5,5) \approx 45$$

$$R(6,6) = ?$$

Exercise & Claim. (multichromatic ramsey theory)

$$\mathcal{R}(3;3,3,3) = 17$$

Fixate one vertex, there are 16 adjacent edges.

There is one color (*with* constraining generality), which occurs at least 6 times. Consider 6 vertices. Cases:

1. There is one red edge between 2 end points. Thus there is a red triangle.
2. There is no red edge. Thus only edges in blue/green paths $R(3,3) = 6$ there is a one-colored triangle.

Exercise.

$$R(3;3,3,3) \geq 17$$

The following example is a K_{16} with 3 colors without a monochromatic triangle.

$$\mathcal{F}_p n, \text{ irreducible polynomial } f \text{ over } \mathcal{F}_p \sigma \mod n$$

$$\mathcal{F}_p[x]/f = \left\{a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} : a_i \in \mathcal{F}_p\right\}$$

$$\text{here } \mathcal{F}_{2^n}/\langle f \rangle = \left\{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_i \in \mathcal{F}_2\right\}$$

For example $x^4 + x + 1$ is irreducible. There are no roots in $\mathcal{F}_2$.

$$x^1, x^2, x^3, x^4 = x + 1$$
$$x^5 = x^2 + x$$
$$x^6 = x^3 + x^2$$
$$x^7 = x^4 + x^3 = x^3 + x + 1$$
$$x^8 = x^4 + x^2 + x = x^2 + 1$$
$$x^9 = x^3 + x$$
$$x^{10} = x^4 + x^2 = x^2 + x + 1$$
$$x^{11} = x^3 + x^2 + x$$
$$x^{12} = x^4 + x^3 + x^2 = x^3 + x^2 + x + 1$$
$$x^{13} = x^4 + x^3 + x^2 + x = x^3 + x^2 + 1$$
$$x^{14} = x^4 + x^3 + x = x^3 + 1$$
$$x^{15} = x^4 + x = \mathbf{1}$$

class $0 x^0, x^0 = x^{15} = 1, x^3, x^6 = x^3 + x^2, x^9 = \ldots, x^{12}$
class $1 x^1, x^4 = x + 1, x^7 = \ldots, x^{10}, \ldots, x^{13}$
class $2 x^2, x^5 = x^2 + x, x^8, x^{11} = \ldots x^{14}$

Edge (v_i, v_j) is colored according to $x^i - x^j = x^k$ $(0 \leq k \leq 14)$. $k = 0 \mod 3$ is color 0 (red). $k = 1 \mod 3$ is color 1 (blue). $k = 2 \mod 3$ (green).

We still have to show that no monochromatic triangles are contained.

$$\text{monochromatic triangle at } (x^i, x^j, x^s)$$
$$(0, x^j - x^i, x^s - x^i)$$
$$(0, 1, \frac{x^s - x^i}{x^k})$$

Exercise.

$$41 \leq R(4; 3, 3, 3, 3) \leq 66$$

$$\left\{x^{4k} \mod 41\right\} = \{1, 4, 10, 16, 18, 23, 25, 31, 37, 40\}$$

Difference 1 must occur in any other case.

$$R(r+1; \underbrace{3, 3, \ldots, 3}_{r+1}) \leq (r+1)\left(R(r; \underbrace{3, \ldots, 3}_{r}) - 1\right) + 2$$

$$R(r; \underbrace{3, \ldots, 3}_{r}) \leq [r!e] + 1$$

$$r = 2 : 6 = [2!e] + 1$$

$$r = 3 : 17 = [3!e] + 1$$

Induction:

$$R(r+1; 3 \ldots) \leq (r+1)[r!e] + 2 = [(r+1)!e] + 1$$

Date. 14.01.07

9 Ramsey Theory with graphs

9.1 Remarks to the proof of the theorem of Turán

Is $n \neq r$. G has no K_r, but maximum number of edges. Due to maximality there must be a K_{r-1} subgraph. Let A_{r-1} be such an $(r-1)$ clique (thus the $r-1$ vertices). $B = V \setminus A_{r-1}$ and $|B| = b - (r-1)$.

Idea: Count edges e_A (edges in A), e_B (edges in B) and e_{AB} (edges from A to B).

$$|e_A| \leq \binom{r-1}{2}$$

B does not contain a K_r. Apply induction hypothesis for $n - (r-1)$.

$$|e_B| \leq (1 - \frac{1}{r-1})(n - (r-1))^2$$

Each edge $b \in B$ is connected to at most $r-2$ edges of A_{r-1} (otherwise there would be a r clique).

$$|e_{AB}| \leq (n - (r-1))(r-2)$$
$$|E| \leq |e_A| + |e_B| + |e_{AB}|$$
$$|E| \leq \frac{1}{2}\left(1 - \frac{1}{r-1}\right)n^2$$

9.2 Second approach for proof

We put n vertices in buckets of size $r-1$. Therefore each bucket contains $\frac{n}{r-1}$ elements. To reach the maximum number of K_{r-1} but no K_r we connect every vertex with every vertex which is not part of the same bucket. Each vertex is connected to $(n - \frac{n}{r-1})$ vertices. Thus the total number of edges is $\frac{n}{2}(n - \frac{n}{r-1})$. QED.

10 Jensen's inequality

Date. 15th of January 2014

$$f(z) = \binom{z}{t} = \frac{z(z-1)(z-2)\dots(z-t+1)}{t!} \qquad \text{convex}$$

$$\sum \binom{d_i}{t} \geq m \binom{\frac{|E|}{m}}{t}$$

$$(s-1)n^t \geq m(\frac{E}{m} - t + 1)^t$$

$$\binom{n}{t} = \frac{n(n-1)\dots(n-t+1)}{t} \leq \frac{n^t}{t!}$$

$$(s-1)^{\frac{1}{t}} n (m^{-1})^{\frac{1}{t}} \geq \frac{E}{m} - t + 1$$