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Feb–April 2013

Contents

1 What is a combinatorial game?

- Blue (Left) and Right (Red) players alternate (ie. 2 players).
- First player—who cannot move—loses.
- Game cannot end in a tie (draw).
- No randomization (dice, cards).
- Each game consists of a finite number of moves.

An impartial game is one where the set of possible moves does not depend on the identity of the player (eg. in Chess only white is allowed to move white pieces - ergo Chess is a partisan game). Technically we have two criteria:

- Every left move is also a right move (and vice versa)
- Every move is impartial

Typical research questions:

- Who wins if Left/Right goes first?
- Is there an optimal strategy?
- Is a game X a combinatorial/impartial game?

Go terminology:

to have Sente You have initiated a move and force opponent to respond accordingly (meaning you take over the lead in the game).

to have Gote Opposite of Sente.

2 Games

Cutcake Given is a rectangle with horizontal or vertical lines (splitting the rectangle into units in rows and columns). In one move Left can cut vertically, Right horizontally. So the units become smaller and smaller and sometime nobody can move (if all units are of size 1).

Hackenbush A set of lines in colors (for any of both players) is given. A move consists of eliminating one line of your color. If a set of connected lines is not tied to the ground any longer, the rest of the lines floats away. Extension: Green lines can be played by any player.

Run, Shove, Run over A sequence of squares is drawn on a paper. Some squares contain either a penny or a dime. Both players can only push coins to the left side. A coin pushing from the most-left square to the left, is off the game. A pushing move moves a coin to the next left position. If there is any connected sequence of coins already at the next left position, all coins of this sequence will be pushed as well. In shove, every coin to the left of the played coin is moved one square left. A run over consists of moving one position to the left and removing the coin current at this position.

Dots and boxes Given is a grid where each intersection is marked with a dot. In one move the player draws a line between two neighboring dots. Once there is a connected line of one color, the number of squares in that area is assigned to the corresponding player. The player with the highest number of squares wins.

Ski Jumps or w/ Jumps There are two lines each assigned to one player. The left player has coins at the left-most position. The right player has coins the right-most position. In each move the player can move one position to the left (Right) or right (Left). With jumps, the player can jump to an empty line 2-above (at the same column) iff the opponent has a coin in the same column one row above. Same applies for below.

NIM Given are several piles of coins. Each player can remove any number of coins of one pile in one move. If no coins are left, the player has lost the game.

Toads and Frogs Toad (Blue) and Frogs (Red) have coins aligned in a sequence of boxes (not necessarily all boxes have one coin in it). Toads move only to the right. Frogs only move to the left. If a coin of an opponent is in front of you and the box behind the opponent is free, you can jump over the opponent. Once all Toads are at the right-most positions and all Frogs are at the left-most, the next player has lost the game.

Fair shares and varied pairs Given are a number of piles of coins. Two moves are possible:

- You can take a set of piles of coins. Split it into a number of equally-sized piles of coins.
- You can take two piles of coins and combine them if and only if the two piles had the same size.

The person which has to move but all piles have size 1 loses. This game might consist of indefinitely many moves.

A game of Cutcake inverted means changing rows and columns of the square. In Hackenbush the inverse game has all red and blue colors swapped.

3 Theory & Math

3.1 Grundy value

We always assume best play*: Every player plays the optimal move to win the game!* Interpretation of the Grundy value:

$G = 0$ Second player wins

$G < 0$ Right wins

$G \parallel 0$ First player wins ("G is fuzzy with zero")

$G > 0$ Left wins

So great numbers are good for Left. Negative values are good for Right. Comparisons are possible:

$$G \leq H$$

H is at least as good for Left as G.

3.1.1 Hint about evaluation

Let's assume we want to evaluate a game's Grundy value. We can evaluate this by comparison with other games we know the Grundy values of. Because > 0 and < 0 does not state the particular Grundy value, we always compare it with other games such that it equals to zero.

$$G - H \stackrel{?}{=} I \Rightarrow G - H - I = 0$$

3.2 Notation

We can rate a game with a Grundy value.

$$G = 4$$

$$H = 0$$

Then we can apply addition and subtraction:

$$G = I$$

$$G - H = 0$$

$$G + (-H) = 0$$

3.3 Tweedle-dum Tweedle-dee

$$G = G$$

$$G - G = 0$$

Once a symmetrical situation occurs, the first player moving loses. Therefore it should be one's goal to provide the opponent a symmetrical gaming situation.

3.4 Curly brackets notation

$\{$left move option 1, left move option 2, $\ldots$ $|$ left move option 1, left move option 2, $\ldots\}$

$$G = \{0, 1 \mid -3\}$$

$$G = \{G^L | G^R\}$$

$$-G = \{-G^R | -G^L\}$$

In G Left can either play a move towards 0 or an other move towards 1. If it is Right's turn there is only a move leaving the opponent with Grundy value 3. If it is Left's turn, Left plays the highest number. So we can strike out any but the greatest number for Left and the smallest number for Right ("Simplification")

3.4.1 Relation between Grundy values and Curly brackets notation

$$2 = \{1 \mid \}, \quad 1 = \{0 \mid \}, \ldots \qquad 0 = \{ \mid \}, \ldots \qquad -1 = \{ \mid 0\}, \quad -2 = \{ \mid -1\} \ldots$$

$$\frac{1}{2} = \{0 \mid 1\}$$

$$\frac{1}{4} = \{0 \mid \frac{1}{2}\}$$

$$\{\frac{1}{2} \mid \} = 1 = \{\frac{1}{2} \mid n\} (\text{with } n > 1)$$

Any game where you have only negative numbers on the left side and positive numbers on the right side, has a Grundy value of zero.
In general you have a game like this:

$$\{a \mid b\} \qquad a, b \in \mathbb{Z}$$

This game represented as a Grundy value x satisfies the following condition:

$$\min(a, b) < x < \max(a, b)$$

You can evaluate x using simplicity. You have to apply the following criteria:

- The denominator of x as small as possible.
- x is as close to zero ("the simplest number of all") as possible.
- x is a dyadic rational (rational number with structure $\frac{a}{2^b}$).

Examples:

$$\{-3 \mid 1\} = 0$$

$$\{1 \mid 3\frac{7}{8}\} = 2$$

3.4.2 Games that are not numbers

Some games cannot be represented with Grundy values.

Star One of them is a Hackenbush game G where you have exactly one green line. One green line means whoever goes first wins. So it's fuzzy with zero.

$$G = \star$$

$$\star \parallel 0$$

$$\star = \{0 \mid 0\}$$

We can also derive:

$$\star + \star = 0$$

$$-\star = \star$$

Up The Up-Game is defined by

$$\uparrow = \{0 \mid \star\}$$
$$0 < \uparrow < \frac{1}{2^n}$$
$$\downarrow = -\uparrow$$
$$-\uparrow = \{\star \mid 0\}$$
$$\{\uparrow \mid 2\} = 1$$

A game of $\frac{1}{2^n}$ is a game of Hackenbush where we have one blue line at the ground and infinitely many red lines on top.

Omega A game with Omega is used to represent an infinite game:

$$G = \omega = \{0, 1, 2, 3, 4, \ldots \mid \} \qquad \text{“infinite”}$$
$$G = Y_\omega = \{0 \mid \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{6}, \ldots\} \qquad \text{“infinitesimal”}$$

To work with an irrational number x, put all numbers below x on the left side and all numbers greater x on the right side.
Actually Omega helps us to represent any surreal number. A hackenbush game of ω is the same like a sequence of blue sticks stacked onto another and the sticks become infinitely small.

3.5 Nimber theory

3.5.1 Grundy theorem

If G is an impartial game, $G = *n$ for some n (Grundy value of G). In other words, the Sprague-Grundy theorem states that a situation in a combinatorial, impartial game in normal play convention can be represented as a single number which is equivalent to the game situation.

3.5.2 Reversible move

$$G = \{a, b, c, \ldots \mid e, f, g, \ldots\}$$

One move of Left (such as a) is reversible if there is a right move of this left move a (a^R) with $a^R \leq G$ in which case replace a by the left moves of a^R. The same applies to the other side: $e \Rightarrow e^L \Rightarrow e^L \geq G$.
We want to look at the game $G = \uparrow + \star$.

$$\uparrow + \star$$
$$= \{0 \mid *\} + \{0 \mid 0\}$$
$$= \{*, \uparrow \mid 0, \uparrow\}$$

The second line is a result of evaluating the next move. For example if Left moves to zero of the left summand, $\star$ is left for the right player. Therefore $\star$ is part of the last line. This can be simplified to the following:

$$\{0, \star \mid 0\}$$

Reversible moves and dominated moves (simplicity) are our two tools to reduce a game to its simplest representation; the “canonical form”.

3.5.3 Nimber definition

Nimber definition: A nimber equal to zero is "First player loses". Nimber different to zero is "First player wins". There is no move from nimber zero to nimber zero.

$$\{*0, *1, *2, *4, *8\} = \{*0, *1, *2\} = *3 \qquad \text{"Dominated moves"}$$

$$\uparrow + * = \{0, * \mid 0\}$$

$$\uparrow * + \uparrow * = \uparrow + \uparrow + * + * = \uparrow + \uparrow = \Uparrow$$

$$\uparrow + * \overset{\text{abbr}}{=} \uparrow *$$

$$\uparrow + * = \{*, \uparrow \mid 0, \uparrow\}$$

$\uparrow$ reverses 0.

3.6 XOR rule

To get the nimber of several piles of coins, take the XOR sum (binary operation) of all piles. This XOR sum is the overall nimber.
To win a game with several piles, reduce a stack in such a way, that the XOR sum becomes zero.

3.7 MEX rule

The MEX ("Minimal excluded") of a set is the smallest ordinal (including zero) not part of the set provided as parameter. Example:

$$\text{mex}(\{0, 1, 2, 4, 8\}) = 3$$

The MEX rule states that if you have several moves as options (leading to nimbers a, b, c), your current situation can be represented as nimber $\text{mex}(\{a, b, c\})$.

$$*n = \{*0, *1, \ldots, *n-1 \mid *0, *1, \ldots, *n-1\}$$

3.8 Another approach to the XOR rule

$$\begin{aligned} & *2 + *3 \\ &= \{*0, *1\} + \{*0, *1, *2\} \\ &= \{*0 + *3, *1 + *3, *0 + *2, *1 + *2, *2 + *2\} \\ &= \{*3, *2, *2, *3, *0\} \\ &= *1 \end{aligned}$$

3.9 Mean value theorem

Given is any game G, there is a number $m(G)$ with

$$-K \leq \underbrace{G + G + \ldots + G}_{\text{n times}} - n \cdot m(G) \leq K \qquad \text{with K independent of n}$$

Ensures that $m(G)$ is not a bad approximation for G (for a class of games not the "optimal" but a "not so bad" move is our target).

3.10 Atomic weights

$$\int^{3} * = \{0+3|0-3\} \qquad \text{"Hot game" - heated by 3}$$

S. Norton: Any game G is

$$x + \sum \int_{e_i}^{t_i} \qquad \text{Sum of infinitesimals}$$

A game is called "all small" if and only if the opponent has a move for any move of yours.
g is all small, then there is a game G computable from

$$\downarrow * + \star \leq g - G\cdot \uparrow \leq \uparrow * + \star$$

4 Solutions to the quizzes

4.1 Solutions Week 1 Extra problems

1. Right loses
2. First player loses
3. Right loses
4. First player loses
5. Left loses

4.2 Solutions Week 2 Quiz

1. A+B: Left wins.
2. C+F: Second player wins.
3. E+F: Right wins.
4. D+E+F: Left wins.

4.3 Solutions Week 2 Extra problems

1. $A = 1$ (Left wins)
2. $B = \frac{1}{2}$ (Left wins)
3. $C = \frac{1}{8}$ (Left wins)
4. $D \| 0$ (First player wins)
5. $E \| 0$ (First player wins)
6. $F = 0$ (Second player wins)
7. $G = 0$ (Second player wins)

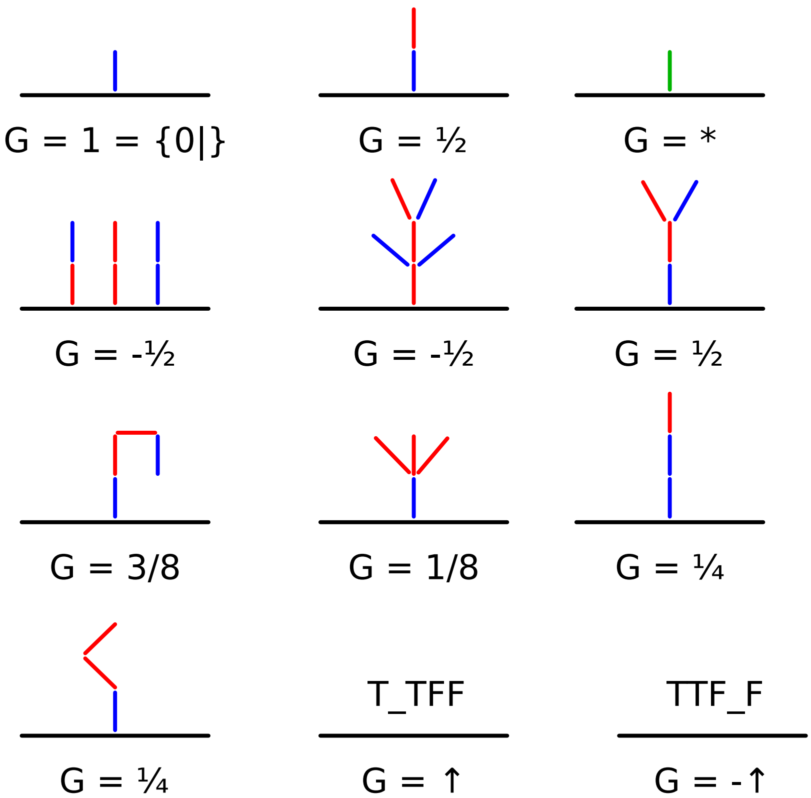

Figure 1: Games as numbers

8. $H < 1\frac{1}{2} =?$ (Right wins)

9. I (Right wins)

10. J (Right wins)

11. K (Right wins)

4.4 Solutions Week 3 Quiz

1. Left first, Left wins. Right first, Right wins ($A \parallel 0$)

2. Left first, Left wins. Right first, Right wins ($B \parallel 0$)

3. Left first, Right wins. Right first, Right wins ($C < 0$)

4. $\{0|*\} + \{0|0\}$ is Left first, Left wins (taking the most-left 0) and Right first, Right wins (taking the most-right 0) ($D \parallel 0$)

4.5 Solutions Week 4 Quiz

1. See figure 4.5.

2. $\{-4| -1\} = -2$

3. $\left\{1|1\frac{3}{4}\right\} = 1\frac{1}{2}$

4. $\left\{0|2\frac{3}{4}\right\} = 1$

5. $\left\{-1|\frac{3}{4}\right\} + 3 = 3$

4.6 Solutions Week 5 Quiz

1. $\{2, 2\frac{1}{2}|3, 6\} = \{2\frac{1}{2}|3\} = 2\frac{3}{4}$

2. $\{0, \uparrow|1\} = \{0|1\} = \frac{1}{2}$

3. $\{\uparrow|2\} = \{0|2\} = 1$

4. $\{\uparrow|0\} = \{0|0\} = *$

5. FT_F = First Player wins = $\{0|0\} = *$

4.7 Solutions Week 6 Quiz

1. $1 \oplus 5 \oplus 11 = 15$

2. 6

3. G(6): $6-1 = 5 = *2, 6-2 = 4 = *1, 6-5 = 1 = *1$. $\text{mex}(\{*2, *1, *1\}) = *0$
G(7): $7-1 = 6 = *0, 7-2 = 5 = *2, 7-5 = 2 = *2$. $\text{mex}(\{*0, *2, *2\}) = *1$

4. G(0): $0 = *0$
G(1): $1 = *1$
G(2): $2 = (1, 0) = (*1, *0) \Rightarrow \text{mex} = *2$
G(3): $1 = (2, 1, 0) = (*2, *1, *0) \Rightarrow \text{mex} = *3$
G(4): $1 = (3, 2, 1) = (*3, *2, *1) \Rightarrow \text{mex} = *0$

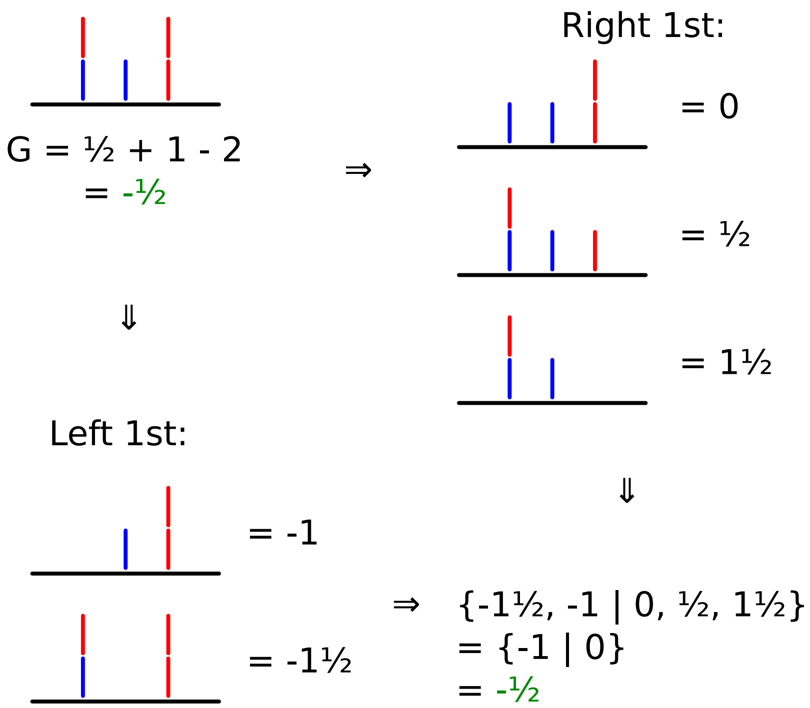

Figure 2: Week 4 Problem 1

4.8 Solutions Week 7 Bonus Quiz

1. $G > 0$
2. G(0): $*0$
   G(1): $*1$
   G(2): $*2$
   G(3): $(1, 2) = (*1, *2) = *0$
   G(4): $(0, 2, 3) = (*0, *2, *0) = *1$
   G(5): $(4, 3, 1) = (*1, *0, *1) = *2$
3. $\{5|5\}$
4. $\frac{3}{4}$
5. XOR(2, 5, 6) $= 1$