Exercise collection for mathematical proofs

Lukas Prokop

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1 Complete Induction

Show by strong induction for all $n \in \mathbb{N}$:

1.1 Equations

$$\sum_{k=1}^{n} k = \frac{n(n-1)}{2} \tag{a}$$

$$\sum_{k=0}^{n} q^k = \frac{q^n - 1}{q - 1} \tag{b}$$

$$\sum_{k=1}^{n} k(k-1) = \frac{1}{3}n(n^2-1) \tag{c}$$

$$\sum_{k=1}^{n} \frac{1}{4k^2-1} = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \tag{d}$$

For $a_k \in \mathbb{N}$:

$$\sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{1}{a_k} \geq n^2 \tag{e}$$

$$\sum_{l=0}^{n} \binom{n}{l} = 2^n \tag{f}$$

$$\sum_{l=0}^{n} (-1)^l \binom{n}{l} = 0 \tag{g}$$

$$\sum_{l=0}^{n} l \binom{n}{l} = n \cdot 2^{n-1} \tag{h}$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \tag{i}$$

1.2 Inequations

$$2^n > n \tag{j}$$

$$\sum_{k=1}^{n} (k-1)^2 < \frac{n^3}{3} \tag{k}$$

For $n \geq 9$:

$$2^{2n} \leq n! \tag{l}$$

For $n \geq 4$:

$$3^n > n^3 \tag{m}$$

For $n \geq 5$:

$$2^n > n^2 \tag{n}$$

1.3 Recursive sequence

Proof the following statement…

$$a_n = 2 + \frac{1}{2^{n+1} - 1} \tag{o}$$

…for the following recursive defined sequence:

$$a_0 = 3; a_n = 3 - \frac{2}{a_n - 1} \quad \text{(p)}$$

2 Proof by contradiction

Meaning: $\sqrt{2}$ is irrational and not rational.

$$\sqrt{2} \in \mathbb{I} \ni \mathbb{Q} \quad \text{(q)}$$

Meaning: $\sqrt{5}$ is irrational and not rational.

$$\sqrt{5} \in \mathbb{I} \ni \mathbb{Q} \quad \text{(r)}$$

$$\text{The number of prime numbers is infinite} \quad \text{(s)}$$